Boolean Algebra

Why study Boolean Algebra?

It is highly desirable to find the simplest circuit implementation with the smallest number of gates or wires.

We can use *Boolean minimization* process to reduce a Boolean function (expression) to its simplest form: The result is an expression with the fewest literals and thus less wires in the final gate implementation.

Boolean Algebra (continued)

- George Boole (1815-1864), a mathematician introduced a systematic treatment of logic.
- He developed a consistent set of postulates that were sufficient to define a new type of algebra: Boolean Algebra (similar to Linear Algebra)
- Many of the rules are the same as the ones in Linear Algebra.

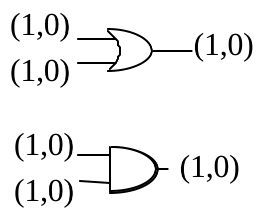
Laws of Boolean Algebra

- There are 6 fundamental laws, or axioms, used to formulate various algebraic structures:

1. *Closure:* Boolean algebra operates over a field of numbers, $B = \{0,1\}$:

   For every x, y in B:

- $x + y$ is in B
- $x \cdot y$ is in B

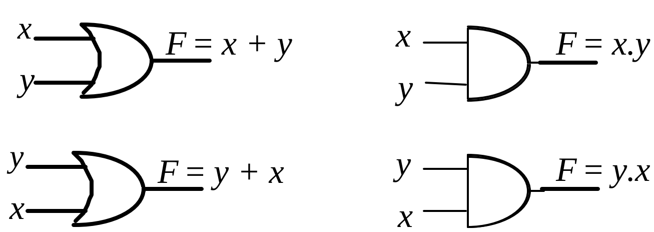


Laws of Boolean Algebra (continued)

2. *Commutative* laws: For every x, y in B,

- $x + y = y + x$
- $x \cdot y = y \cdot x$

» Similar to Linear Algebra

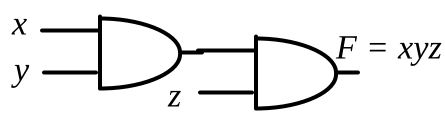


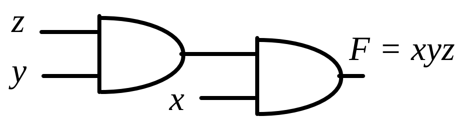
Laws of Boolean Algebra (continued)

3. Associative laws: For every x, y, z in B,

- $(x + y) + z = x + (y + z) = x + y + z$
- $(xy)z = x(yz) = xyz$

» Similar to Linear Algebra





Laws of Boolean Algebra (continued)

4. *Distributive laws*: For every x, y, z in B,

- $x + (y.z) = (x + y)(x + z)$ [+ is distributive over **.**]

» NOT Similar to Linear Algebra

- $x.(y + z) = (x.y) + (x.z)$ [**.** is distributive over +]

» Similar to Linear Algebra

Laws of Boolean Algebra (continued)

5. Identity laws:

- A set *B* is said to have an identity element with respect to a binary operation {.} on *B* if there exists an element designated by 1 in *B* with the property: <u>$1 . x = x$</u>
  Example: AND operation
- A set *B* is said to have an identity element with respect to a binary operation {+} on *B* if there exists an element designated by 0 in *B* with the property: <u>$0 + x = x$</u>
  Example: OR operation

» Similar to Linear Algebra

Laws of Boolean Algebra (continued)

6. Complement

For each x in B, there exists an element x' in B (the complement of x) such that:

- $x + x' = 1$
- $x . x' = 0$

» Similar to Linear Algebra

We can also use $\overline{x}$ to represent complement.

Laws of Boolean Algebra (Summary)

Commutative

$x + y = y + x \qquad xy = yx$

Associative

$(x + y) + z = x + (y + z)$

$(xy)z = x(yz)$

Distributive

$x + (yz) = (x + y)(x + z)$

$x(y + z) = (xy) + (xz)$

Identity

$x + 0 = x \qquad x \,.\, 1 = x$

Complement

$x + \overline{x} = 1 \qquad x \,.\, \overline{x} = 0$

OR *with 1* **AND *with 0***

$x + 1 = 1 \qquad x \,.\, 0 = 0$

Other Theorems

- Theorem 1(a):
  $x + x = x$

$$x + x = x$$

$$\begin{aligned} x + x &= (x + x).1 \\ &= (x + x)(x + x') \\ &= xx + xx' \\ &= x + 0 \\ &= x \end{aligned}$$

- Theorem 1(b):
  $x \,.\, x = x$

$$x \,.\, x = x$$

$$\begin{aligned} x \,.\, x &= xx + 0 \\ &= xx + xx' \\ &= x(x + x') \\ &= x\,.1 \\ &= x \end{aligned}$$

Other Theorems (continued)

- Theorem 2(a):

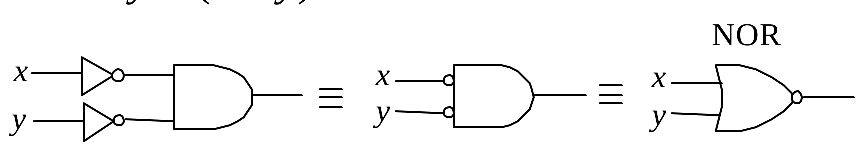
$$
\begin{aligned}
x+1 &= 1 \\
x+1 &= 1.(x+1) \\
&= (x+x')(x+1) \\
&= xx+x'.1 \\
&= x+x' \\
&= 1
\end{aligned}
$$

- Theorem 2(b):

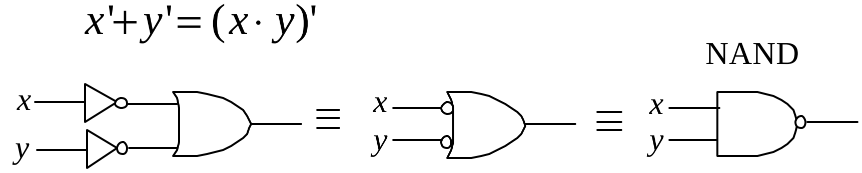
$$
\begin{aligned}
x+xy &= x \\
x+xy &= x(1+y) \\
&= x(y+1) \\
&= x.1 \\
&= x
\end{aligned}
$$

Gate Equivalency and DeMorgan's Law

$$x'\cdot y' = (x+y)'$$



$$x'+y' = (x\cdot y)'$$



Digital Logic Q's & A's

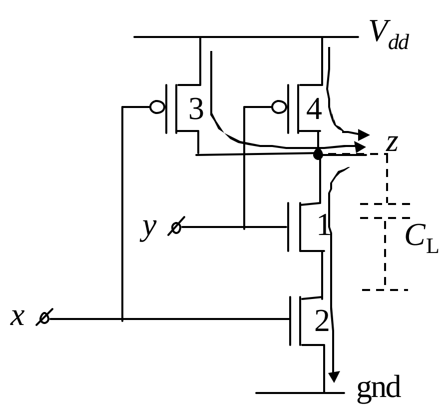
Q: Why is Gate Equivalency useful?

A: It allows us to build functions using only one gate type.

Q: Why are digital circuits constructed with NAND/NOR rather than with AND/OR?

A: NAND and NOR gates are smaller, faster, and easier to fabricate with electronic components. <u>They are the basic gates used in all IC digital logic</u>.

Digital IC's – Transistor Level



x or y: 'low'
transistor 1 or 2 is OFF
transistor 3 or 4 is ON

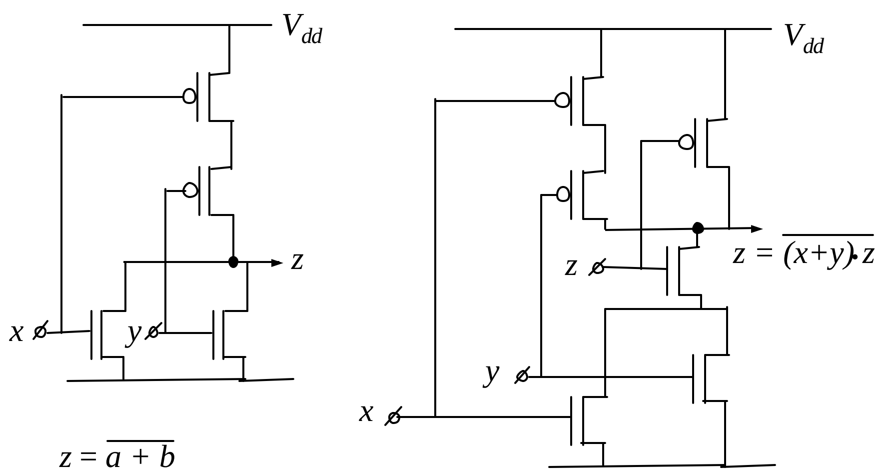
➡ z = 'high'

x and y: 'high'
transistor 1 and 2 are ON
transistor 3 and 4 are OFF

➡ z = 'low'

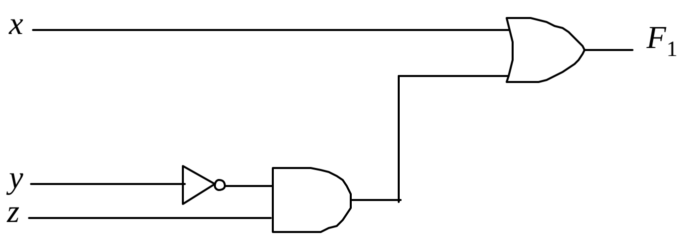
$z = \overline{x \cdot y}$

Digital IC’s (continued)



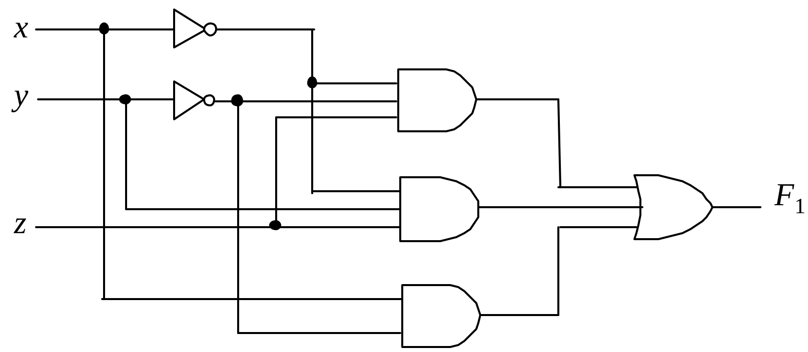
Implementation of Boolean Functions

Example 1: $F_1 = x + y' z$



Implementation of Boolean Functions

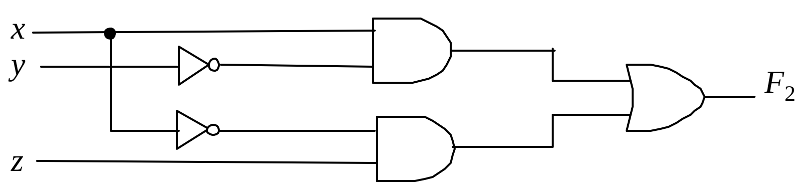
Example 2: $F_1 = x'y'z + x'yz + xy'$



Implementation of Boolean Functions

- Try another implementation using a simplified F_2:

$$\begin{aligned} F_2 &= x'y'z + x'yz + xy' \\ &= x'z(y'+y) + xy' \\ &= x'z(1) + xy' \\ &= x'z + xy' \end{aligned}$$



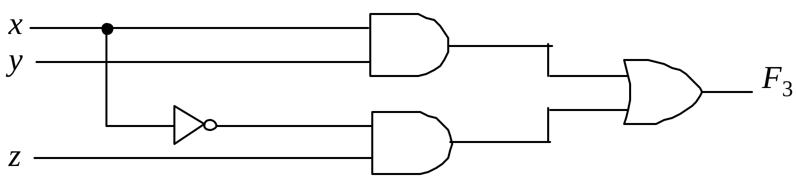
What are the advantages of this implementation?

This implementation has fewer gates and fewer inputs to the gates (or wires) than the previous one.

Simplifying Boolean Functions

- Simplify the following Boolean function to a minimum number of terms: $F_3 = xy + x'z + yz$

$$\begin{aligned} F_3 &= xy + x'z + yz \\ &= xy + x'z + yz(x + x') \\ &= xy + x'z + xyz + x'yz \\ &= xy(1+z) + x'z(1+y) \\ &= xy + x'z \end{aligned}$$



More on complements (DeMorgan)

- Find the complement of: $F = (AB'+C)D'+E$

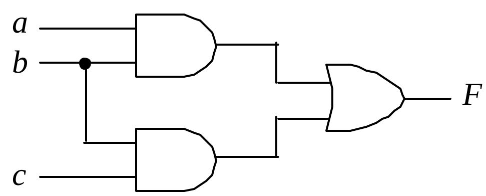
$$\begin{aligned} F' &= [(AB'+C)D'+E]' \\ &= [(AB'+C)D']'E' \\ &= [(AB'+C)'+D'']E' \\ &= [(AB')'C'+D]E' \\ &= (A'+B)C'E'+DE' \end{aligned}$$

- Show that the complement of $x(x+y) = x'$

$$\begin{aligned} [x(x+y)]' &= x'+(x+y)' \\ &= x'+x'y' \\ &= x'(1+y') \\ &= x'(1) = x' \end{aligned}$$

Implementation of Boolean Functions

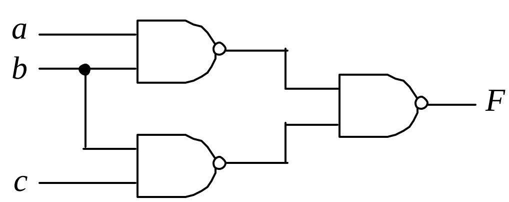
▪ Draw the logic diagram for the following function: $F = (a.b)+(b.c)$



Implementation of Boolean Functions

▪ Using <u>ONLY NAND</u> gates, draw a schematic for the following function: $F = (a.b)+(b.c)$

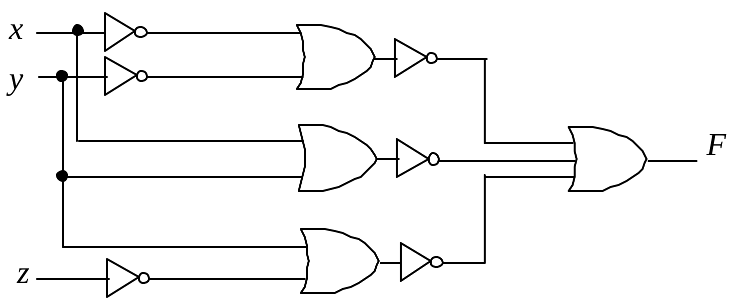
$$(F')' = [[(a.b)+(b.c)]']'$$
$$= [(a.b)'.(b.c)']'$$



Implementation of Boolean Functions

▪ Using only OR and NOT gates, draw a schematic for the following function: $F = xy + x'y' + y'z$

$$(F')' = ((xy + x'y' + y'z)')'$$
$$= [(xy)'.(x'y')'.(y'z)']'$$
$$= [(x'+y').(x+y).(y+z')]'$$
$$= (x'+y')' + (x+y)' + (y+z')'$$



Minterms and Maxterms

➢ MINTERMS AND MAXTERMS:

n binary variables can be combined to form 2^n terms (AND terms), called *minterms* (SOP).

In a similar fashion, *n* binary variables can be combined to form 2^n terms (OR terms), called *maxterms* (POS).

* Note that each maxterm is the complement of its corresponding minterm and vice versa.

Minterms and Maxterms (continued)

Table 2-3:
Minterms and Maxterms for Three Binary Variables

x y z	**minterms**		**Maxterms**	
0 0 0	$x'y'z'$	m_o	$x+y+z$	M_o
0 0 1	$x'y'z$	m_1	$x+y+z'$	M_1
0 1 0	$x'yz'$	m_2	$x+y'+z$	M_2
0 1 1	$x'yz$	m_3	$x+y'+z'$	M_3
1 0 0	$xy'z'$	m_4	$x'+y+z$	M_4
1 0 1	$xy'z$	m_5	$x'+y+z'$	M_5
1 1 0	xyz'	m_6	$x'+y'+z$	M_6
1 1 1	xyz	m_7	$x'+y'+z'$	M_7

Sminterms and Pmaxterms

- Given the truth table, express F_1 in sum of minterms

x	y	z	F_1	F_2
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

$$F_1(x, y, z) = \sum(1,4,5,6,7) = m_1 + m_4 + m_5 + m_6 + m_7$$

$$= (x'y'z) + (xy'z') + (xy'z) + (xyz') + (xyz)$$

- Find F_2

S minterms and P maxterms

- Repeat for product of maxterms.

x	y	z	F_1	F_2
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

$$F_1(x,y,z) = \prod(0,2,3) = M_0 \cdot M_2 \cdot M_3$$

$$= (x+y+z)(x+y'+z)(x+y'+z')$$

S minterms and P maxterms

Express the Boolean function $F = x + y'z$ in a sum of minterms, and then in a product of Maxterms.

$$x = x(y+y') = xy + xy'$$

$$xy = xy(z+z') = xyz + xyz'$$

$$xy' = xy'(z+z') = xy'z + xy'z'$$

$$y'z = y'z(x+x') = xy'z + x'y'z$$

Adding all terms and excluding recurring terms:

$$F(x,y,z) = x'y'z + xy'z' + xy'z + xyz' + xyz \quad \text{(SOP)}$$

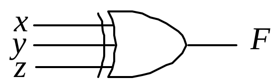
$$F(x,y,z) = m_1 + m_4 + m_5 + m_6 + m_7 = \sum(1,4,5,6,7)$$

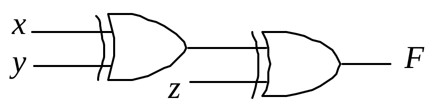
Product of maxterms (POS)? $\prod(0,2,3) = M_0 \cdot M_2 \cdot M_3$

XOR Logic gate

3-input exclusive-OR (XOR) logic gate:

$F = X \oplus Y \oplus Z$





X	Y	Z	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1